The Effects of Electron Scattering on Čerenkov Light Output
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1 Introduction

The standard derivation of the equations describing the emission of Čerenkov radiation is based on the assumption that the charged particle moves in a straight line at a constant velocity (see, for example, Jelley [1, Chapter 2], Jackson [2] or Schiff [3]). However, this is not a good assumption for electrons travelling in a dense medium such as water, where the innumerable collisions with the atoms cause the electrons to deflect from straight paths and to slow down. An attempt at qualitatively understanding the effects of scattering on the emission of Čerenkov radiation is made. The effects of multiple scattering have been treated by several authors for the case of electrons travelling through a thin transparent plate [4, 5, 6, 7]. Their results are summarized here, and a possible method of extending the calculations over the entire paths of electrons in water is given.

2 The Standard Derivation

Schiff's derivation [3] of the angular distribution for Čerenkov radiation produced by a constant-velocity, straight-moving charge is presented here, as his formalism allows the calculation to (in principle) be extended over more realistic paths in dense media. He starts with the current density for a point charge e which is located at the origin at $t = 0$ and moving in the z direction with constant speed v,

$$J_x(\mathbf{r}, t) = J_y(\mathbf{r}, t) = 0$$

$$J_z(\mathbf{r}, t) = ev\,\delta(x)\,\delta(y)\,\delta(z - vt) \tag{1}$$

To calculate the angular distribution of the Čerenkov radiation, we use the exact expression for the average energy radiated at position $\mathbf{r}$ by a harmonically time-varying current distribution in a homogeneous isotropic dielectric medium

$$P_{k\omega}(\mathbf{r}) = \frac{nk^2}{2\pi r^2 c}\left|\int J_{\perp k}(\mathbf{r}')exp(-in\mathbf{k}\cdot\mathbf{r}')d\tau'\right|^2 \tag{2}$$

where $P_{k\omega}$ is the component of the Poynting vector in the direction of observation (parallel to $\mathbf{k}$ or $\mathbf{r}$), $\mathbf{k}$ has magnitude ω/c, n is the index of refraction for the medium, and $J_{\perp k}$ is the component of the current density perpendicular to $\mathbf{k}$. The expression for the radiated energy (eqn. 2) was developed for harmonically time-dependent current densities, hence, the density in eqn. 1 must be replaced by the Fourier amplitude of angular frequency ω

$$J_{z\omega}(\mathbf{r}) = \frac{e}{2\pi}\,\delta(x)\,\delta(y)\,exp(i\omega z/v) \qquad (3)$$

Thus, $J_{\perp k} = J_{z\omega}\sin\theta$. A little algebra yields the energy flow per unit area and angular frequency

$$2\pi P_{k\omega}(\mathbf{r}) = \frac{ne^2\omega^2\sin^2\theta}{4\pi^2c^3r^2}\left|\int exp\left[i\omega z'\left(\frac{1}{v}-\frac{n\cos\theta}{c}\right)\right]dz'\right|^2 \qquad (4)$$

For a pathlength L centered on the origin, the integral is evaluated to be

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} e^{i\omega z'\left(\frac{1}{v}-\frac{n\cos\theta}{c}\right)}dz' = \frac{2\sin\left[\frac{\omega L}{2}\left(\frac{1}{v}-\frac{n\cos\theta}{c}\right)\right]}{\omega\left(\frac{1}{v}-\frac{n\cos\theta}{c}\right)} \qquad (5)$$

This is just the definition of the delta function in the limit of L going to infinity

$$\lim_{L\to\infty}\frac{2\sin\left[\frac{\omega L}{2}\left(\frac{1}{v}-\frac{n\cos\theta}{c}\right)\right]}{\omega\left(\frac{1}{v}-\frac{n\cos\theta}{c}\right)} = 2\pi\,\delta\left(\frac{1}{v}-\frac{n\cos\theta}{c}\right)$$

from which we get the familiar relation defining the half-angle of the Čerenkov cone

$$\cos\theta_0 = \frac{c}{nv}$$

Combining eqn. 5 with the $\sin^2\theta$ term from eqn. 4 we see that, for a path of finite length L, the shorter distance over which coherence takes place produces an angular distribution of the form

$$\sin^2\theta\,L^2\,\frac{\sin^2\chi}{\chi^2} \quad ; \quad \chi = \frac{\omega L}{2}\left(\frac{1}{v}-\frac{n\cos\theta}{c}\right) \equiv \frac{\pi L}{\lambda'}\left(\frac{1}{n\beta}-\cos\theta\right) \qquad (6)$$

where $\lambda' \equiv 2\pi c/n\omega$ is the wavelength in the medium. The behaviour of eqn. 6 is quite different in the limit of small and large L. For $L \gg \lambda'$, the angular distribution is still sharply peaked at the usual Čerenkov angle θ_0, however, it has a diffractive pattern whose full width at half maximum is

$$\delta\theta \simeq \frac{\lambda'}{L\sin\theta_0}$$

As expected, the peak gets sharper as L increases. However, as the pathlength decreases below the wavelength of the radiation ($L \ll \lambda'$), the $\sin^2\chi/\chi^2$ term becomes constant, and the radiation is emitted over a dipole angular distribution.

To get the total energy radiated over the path L, one integrates eqn. 4 over the surface of a sphere of radius r. The integral can be evaluated easily in the two limits mentioned above. For the case of $L \gg \lambda'$ (and thus $\delta\theta \ll 1$), $\sin\theta$ can be replaced by $\sin\theta_0$, and the limits of integration on $\cos\theta$ can be extended to $\pm\infty$. The integral of eqn. 6 then evaluates to $4\pi c L r^2 \sin^2\theta_0/n\omega$. Substitution of this result into eqn. 4 yields the total energy radiated per unit angular frequency range over the path L

$$\frac{\omega e^2 L \sin^2\theta_0}{c^2} = \frac{\omega e^2 L}{c^2}\left(1 - \frac{1}{n^2\beta^2}\right) \tag{7}$$

This is the standard result, showing that, for constant index of refraction n, the total power output is proportional to ω and to the path length L, and thus, the number of photons emitted per unit frequency interval per unit length is constant. In reality, there is usually a weak frequency dependence brought in through the index of refraction.

In the limit of $L \ll \lambda'$, the integral over the dipole angular distribution yields a total radiated energy per unit angular frequency over the path L of

$$\frac{\omega e^2 L}{c^2}\left(\frac{4L}{3\lambda'}\right) \tag{8}$$

Hence, the number of photons emitted per unit frequency range is not constant as in the other limiting case, but goes as the ratio of L/λ'. This difference may prove useful when looking for evidence of the loss of coherence (see the Conclusions section).

To get an estimate of what pathlength is required to achieve coherence, we look at the result of the integration over the angular distribution (eqn. 6). The value of the integral over $\sin^2\chi/\chi^2$ goes to π as the limits on χ go to $\pm\infty$. About 90% of this maximum value is achieved when the limits on χ are $\pm\pi$. This translates to the condition

$$\left|\frac{L}{\lambda'}(\frac{1}{\beta} \pm n)\right| > 1 \tag{9}$$

which, for β=1 and n=4/3, becomes

$$L > 3\lambda'$$

The longest wavelength we are interested in is $n\lambda' = 720$ nm. Hence, we need straight pathlengths of at least 2 μm to get full light output. Molière's theory predicts the mean distance between multiple scatters to be $t/\Omega_0 = \beta^2/b_c$ (see, for example, the EGS4 manual). For water, $b_c^{-1} \simeq$1.3 μm, which is less than the distance computed from our limit above. However, the multiple scattering is predominately small-angled, so we can expect that coherence between the straight path segments will be preserved to a large extent.

Before we leave the simplest case of a straight path, it is worthwhile to summarize Jelley's report [1, pages 26-30] on the effects of the slowing down of the electron and the radiation reaction from the emitted radiation. In the former case, coherence is said to be preserved if the deceleration of the electron is gradual enough that the following condition holds:

$$T\left(\frac{dv}{dt}\right) \ll \frac{c}{n}$$

where T is the period of the wave. This can be re-expressed as

$$\lambda\left(\frac{dE}{dx}\right)\frac{1}{\gamma^3 mc^2} \ll \frac{1}{n} \tag{10}$$

For our purposes, we can conservatively estimate the quantity on the left hand side of eqn. 10 using $\lambda = 720 \times 10^{-7}$ cm, $(dE/dx) = 3$ MeV/cm and $\gamma = 1.51$ (electron kinetic energy of 261 keV) to be ~0.0001. This is much smaller than $(1/n) = 0.75$ in water. Thus, the slowing down of the electron in the water should not affect the coherence of the Čerenkov radiation. The quantum mechanical effects of the reaction of the Čerenkov radiation on the electron are to introduce terms of order $(\hbar\omega/mc^2)$ into the expression for the energy loss per unit length (eqn. 7). For photon energies of a few eV, these terms are negligible.

3 Multiple Straight Pathlengths

In this section, we will investigate the effect of breaking the electron path into many short straight segments on the coherence of the radiation. The following notation was developed by Dedrick [4], who used it when determining the effects of multiple scattering on the angular distribution of Čerenkov radiation. The integral in eqn. 2 becomes a sum of integrals over each straight path segment ν

$$P_{k\omega}(\mathbf{r}) = \frac{nk^2}{2\pi r^2 c} \left| \sum_{\nu=1}^{N} I_\nu \right|^2 \qquad (11)$$

where

$$I_\nu = \frac{e}{2\pi} \sin\Theta_\nu exp[i\omega t_\nu - ink(x_\nu \sin\theta\cos\phi + y_\nu \sin\theta\sin\phi + z_\nu\cos\theta)] \times \frac{exp[i\omega l_\nu(\frac{1}{v_\nu} - \frac{n}{c}\cos\Theta_\nu)] - 1}{i\omega(\frac{1}{v_\nu} - \frac{n}{c}\cos\Theta_\nu)}$$

and l_ν is the length of the segment from x_ν, y_ν, z_ν to $x_{\nu+1}, y_{\nu+1}, z_{\nu+1}$, θ_ν and ϕ_ν are the polar angles for the segment, and t_ν is the time at which the particle is at the start of the path segment. The emission angle Θ_ν is taken between the path segment and $\mathbf{k}$, and obeys the relation $\cos\Theta_\nu = \cos\theta\cos\theta_\nu + \sin\theta\sin\theta_\nu\cos(\phi - \phi_\nu)$. The integral I_ν can be expressed in a notation similar to that used previously

$$I_\nu = \frac{e}{2\pi} \sin\Theta_\nu \, e^{i(\delta_\nu + \chi_\nu)} \, l_\nu \, \frac{\sin\chi_\nu}{i\chi_\nu} \qquad (12)$$

where the phase angles are given by

$$\chi_\nu = \frac{\omega l_\nu}{2}\left(\frac{1}{v_\nu} - \frac{n}{c}\cos\Theta_\nu\right) \equiv \frac{\pi l_\nu}{\lambda'}\left(\frac{1}{n\beta_\nu} - \cos\Theta_\nu\right)$$

$$\delta_\nu = \omega t_\nu - nk(x_\nu \sin\theta\cos\phi + y_\nu\sin\theta\sin\phi + z_\nu\cos\theta)$$

Diffractive effects over each path segment are determined by the angle χ_ν, while coherence effects between segments are determined by both phase angles.

Some general statements can be made based on the last few equations. Once again we see that the integral I_ν is significant only when χ_ν is of order π or less (ie. when Θ_ν is within $\sim \lambda'/l_\nu$ of the Čerenkov angle θ_0). The expression for the total energy radiated in any direction is composed of terms of the form $I_\nu I_\nu^*$ and $(I_\nu I_\mu^* + I_\nu^* I_\mu)$. The former represent the contribution from each path segment as derived in the previous section, while the latter represent the interference between the segments. The interference terms are only significant where the cones from two segments overlap.

Once again, let us look at what happens in the limiting cases of large and small pathlengths. In the case of long segments ($l_\nu \gg \lambda'$), the diffractive widths of the cones are very small. Thus, unless the cones are aligned and have similar opening angles, the overlap of the two cones will be small (confined to the lines of intersection between the cones). When integrating over all angles to get the total energy radiated, the range of the angles χ_ν will be large. Thus, the energy contribution from each segment will tend toward the maximum value derived in the previous section (cf. eqn. 7). The contribution from each interference term averages to zero as one integrates terms of the form

$$\cos[(\delta_\nu - \delta_\mu) + (\chi_\nu - \chi_\mu)] \frac{\sin \chi_\nu}{\chi_\nu} \frac{\sin \chi_\mu}{\chi_\mu}$$

over large ranges of χ. Thus, long segments can be treated more or less independently. The total light output over each segment is the maximum value derived in the previous section. There will be diffractive effects in the angular distribution at the intersections of the cones, but these are localized to small regions of (θ, ϕ), and average to zero over each region. For those cases where the two cones are aligned and have similar opening angles, the phase angles δ determine whether or not the interference is constructive or destructive. The difference in the phase angles is given by

$$\Delta\delta = (\delta_\nu - \delta_\mu) = \omega(t_\nu - t_\mu) - n(\mathbf{x}_\nu - \mathbf{x}_\mu) \cdot \mathbf{k}$$

For a particle travelling in a straight line, $t_\nu - t_\mu = -|\mathbf{x}_\nu - \mathbf{x}_\mu|/v$ (for $\nu < \mu$), hence,

$$\Delta\delta = -\omega|\mathbf{x}_\nu - \mathbf{x}_\mu| \left(\frac{1}{v} - \frac{n}{c}\cos\Theta\right)$$

which goes to zero at $\Theta = \theta_0$. To get destructive interference, the particle would have to scatter out of and back into the same direction over a path whose length is different from the distance $|\mathbf{x}_\nu - \mathbf{x}_\mu|$ by some half-integral number of wavelengths. It is not unreasonable to believe that such situations will occur. The question is how often, and how destructive is the interference? The answer lies in the details of the problem, and I think, cannot be answered in any handwaving way. A deeper analysis is required, as discussed in the Conclusions of this report.

For the case of $l_\nu \ll \lambda'$, the dipole angular distributions are broad, and interference between segments cannot be ignored. It is this interference, in fact, which produces the Čerenkov cone over long pathlengths. Thus, the problem cannot be simplified by treating segments independently. However, we know from the previous section that the mean free path between scatters for the electron is comparable to the wavelength of the Čerenkov radiation in water, so it is likely that the true situation lies somewhere between the two limiting cases for L.

It should be noted that the expression for total energy radiated (eqn. 11) seems to be in a useful form for analysis using Monte Carlo techniques. In principle, an electron's position, direction of motion, and speed at the start of each step can be stored during a Monte Carlo simulation, and used to calculate the sum of the integrals I_ν afterwards. From this, the total energy radiated and the angular distribution at various wavelengths can be calculated. However, there are problems which make applying these results difficult. The numerical integration over all angles and frequencies required to calculate the total energy output would be very time consuming. Also, the almost discrete nature of the angular distribution when the cones are sharp makes it difficult to efficiently choose a direction for the emission of the photon. And finally, we have so far neglected the multiple scattering of the electron over each step. The last effect is treated in the next section.

4 Multiple Scattering Over a Straight Path Segment

The effects of electron multiple scattering on the angular distribution of Čerenkov light produced in thin transparent plates have been studied by Dedrick [4], and by Kobzev and associates [5, 6, 7]. I shall only briefly report the results, as the derivations are rather involved.

Dedrick starts from the expression for the energy output per unit frequency in terms of the sum of integrals over straight path segments (eqn. 11). Using various approximations to reduce the expression for the integral I_ν to simpler forms for the two cases of photon emission angles close to and far from the Čerenkov angle θ_0, and using a simple multiple scattering theory to calculate averages over the interference terms, he derives angular distributions for photon emission in terms of a reduced angle $\delta = \sqrt{2}(\theta - \theta_0)/\sqrt{\langle\vartheta^2\rangle}$ and a parameter $K = [(6\pi/\sqrt{2})n \sin\theta_0(L/\lambda)\sqrt{\langle\vartheta^2\rangle}]^{\frac{1}{3}}$, where $\sqrt{\langle\vartheta^2\rangle}$ is the rms multiple scattering angle for the plate of thickness L. The results for small reduced angle δ were computed numerically as a function of K, and plotted.

Kobzev *et. al.* [6] were not satisfied with the restrictiveness of some of Dedrick's small angle approximations. They present new formulae for the angular distribution for each photon polarization that involve unevaluated integrals over time. The formulae were derived elsewhere [8], and I have not been able to acquire the translation (if someone has a copy of this paper, please send it to me). Although I reserve my final judgement until I see the actual derivations of their results, the integral form of their equations makes it doubtful that they will be useful for us.

Both treatments of multiple scattering were for electrons travelling through a thin plate, ie. there were definite boundaries for the production of the Čerenkov radiation, and slowing down of the electron was negligible. This makes there analysis somewhat easier than ours, since they don't need to consider interference with radiation produced over other path segments. Unfortunately, their results are in the form of angular distributions, and not amplitudes. Thus, we cannot use their results if we wish to calculate interference effects between path segments over which multiple scattering occurred. It may be possible to derive these from the original papers, although the ensemble averages used in Dedrick's derivation seem to make this unlikely in that case.

5 Conclusions

When an electron travels in water, the average distance between scatters is only slightly larger than the wavelength of the Čerenkov radiation. However, the fact that the scattering is predominately at small angles means that some level of coherence will still be achieved. We need to find out what that level is before we can hope to accurately predict the frequency and angular distributions of the Čerenkov light. It is likely that the level of coherence is high, since past experiments [9, 10, 11] have found reasonable agreement between the measured and predicted wavelength distributions. However, there is marginal evidence in the most sensitive of these experiments [11] that longer wavelengths are slightly suppressed (in my opinion, not the authors'). This is what would be expected if full coherence has not been reached (cf. eqns. 7 and 8). I have not found any reports of accurate measurements of the absolute intensity of Čerenkov radiation, so a direct check of the standard theoretical predictions at the level of a few percent is not possible yet.

Theoretically, the expression for the energy output in terms of the sum over small, straight path segments (eqn. 11) offers some hope that we may be able to calculate the Čerenkov spectrum even if the radiation is not fully coherent. The fly in the ointment is that we cannot use the published data on multiple scattering if we want to accurately compute the effects of interference between the radiation produced by separate segments. Perhaps step sizes of order ~ 1 μm could be used so that multiple scattering would become unimportant, and plural scattering theory could be used instead. Then, no approximations need be made about interference between separate steps. The results of such computer-intensive calculations could then be compared to those done with the various approximations mentioned in this report, to see which are acceptable for use in generating SNO events.

There is one final point I would like to make about our simulations of Čerenkov radiation. The intensity of the radiation is proportional to the pathlength of the electron. Thus, even if the intensity per unit pathlength is precisely given by the standard expression (eqn. 7), the total light output is as uncertain as the pathlength. Since the actual pathlength is estimated when multiple scattering approximations are made, some effort should be put into making sure that these estimates are accurate.

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